Lecture Notes for PHY 405
Classical Mechanics

From Thorton & Marion's *Classical Mechanics*

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Chapter 2: Newtonian Mechanics—Single Particle

Chapters 2-8 will largely cover the dynamics of a single, isolated particle.

We'll examine systems of multiple particles in chapters 9+.

We will also make use of Chapter 1, which contains many useful formulae, namely, how to express a particle's position vector $\mathbf{r}$, its velocity vector $\dot{\mathbf{r}} = d\mathbf{r}/dt$, & acceleration vector $\ddot{\mathbf{r}}$ in a variety of coordinate systems (eg, Cartesian, cylindrical, spherical, etc).

There is also useful stuff in the back of the book:
solutions to problems, tables of integrals, etc.

Newton's Laws

Particles in the classical macroscopic world obey Newton's laws, first published in *Principia*, 1687

Note that a physical law=rule obeyed by nature and confirmed by experiments.

Law I: A body remains at rest or in uniform motion unless acted upon by a force, ie., $\mathbf{v}$ = constant provided $\mathbf{F} = 0$.

Law II: A body acted upon by a force moves such that its time rate of change of momentum equals the force, ie., $\dot{\mathbf{p}} = \mathbf{F}$ where $\mathbf{p} = m\dot{\mathbf{r}}$, where m is the particle's mass, $\mathbf{r}$ its position vector, $\dot{\mathbf{r}} = d\mathbf{r}/dt$ its velocity.

Law III: If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction, ie, $\mathbf{F_{12}} = -\mathbf{F_{21}}$, where $\mathbf{F_{12}}$ = the force on particle 1 exerted by particle 2.

These laws apply in a non–relativistic, macroscopic world, ie., to objects much larger than the de Broglie wavelength $\lambda = h/p$ that are moving slowly compared to the speed of light c.

Note that III implies momentum conservation:

$$\begin{aligned} \mathbf{F_{12}} = \dot{\mathbf{p_1}} &= -\dot{\mathbf{p_2}} \\ \text{so} \quad \dot{\mathbf{p_1}} + \dot{\mathbf{p_2}} &= 0 \\ &\Rightarrow \mathbf{p_{total}} = \mathbf{p_1} + \mathbf{p_2} = \text{ constant} \end{aligned}$$

Reference Frames

Application of Newton's laws requires choosing an *inertial* reference frame, such as an x, y, z coordinate system.

Inertial reference frame=reference frame where Newton's laws are obeyed (a rather circular argument...).

Law I implies that an inertial reference frame is one that is stationary or moves with velocity $v =$ constant.

However there is no reference frame at 'absolute rest' since everything (planets, stars, galaxies, etc) all move relative to each other.

However we only need a reference frame that is 'good enough'

Consider an experiment that involves a pendulum. It dangles it from a table and oscillates at the frequency

$$\omega = \sqrt{\frac{g}{\ell}}$$

which is sensitive to the acceleration due to gravity $g = 9.8$ m/sec^2, where ℓ=pendulum length.

Does the table provide an inertial reference frame?

This depends upon the desired precision of your experiment. Note that the table is actually an accelerated reference frame, since it sits on the Earth $\oplus$ which rotates once/day ($P = 1$ day, $\omega = 2\pi/P = 7.3 \times 10^{-5}$ sec^{-1}, $R_\oplus = 6400$ km)

$$a_{centrifugal} \sim \omega^2 R_\oplus \sim 0.003g$$

$\Rightarrow$the table is a 'good enough' reference frame if only 2 digits of experimental precision is required. Greater precision will require a reference frame that co–rotates with the Earth.

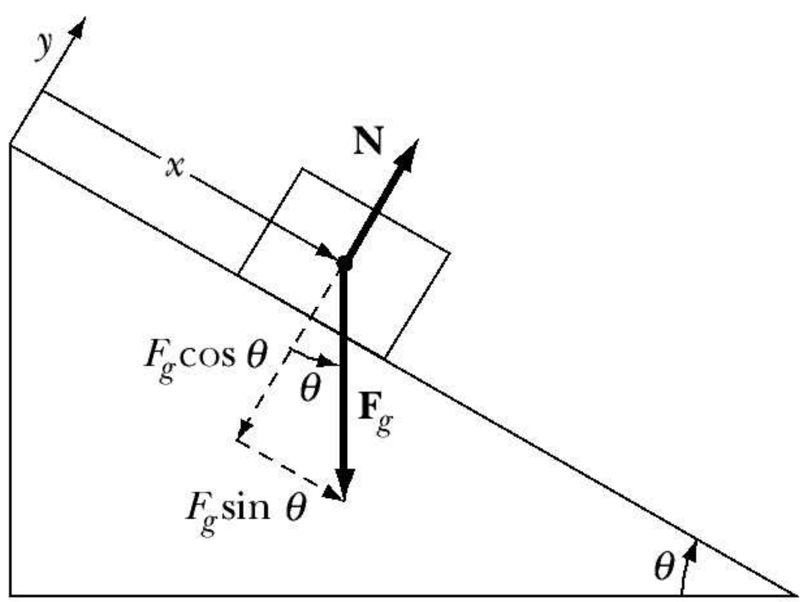
Note that $a_{centrifugal} \sim 0.0006g$ due to $\oplus$'s orbital motion about the Sun, so an experimental accuracy of 4+ digits will require using a reference frame that corotates with the rotating & orbiting Earth.

But the Sun is not at rest—it orbits the Solar System's center–of–mass, which orbits the Galactic Center, etc...

In practice, you only need to choose reference frame that is 'inertial enough', ie., pick a reference frame such that the accelerations due to other external phenomena (due to rotations or other accelerations) are negligible.

Applying Newton's Laws

Do example 2.1 from text: a block slides down a frictionless plane that is inclined by angle θ from the horizontal. (1) What is the block's acceleration? (2) What is its velocity as a function of time? (3) as a function of position?



Step 1. Choose a coordinate system.

Step 2. Asses the forces:

$$\begin{aligned}
\mathbf{N} &= N\hat{\mathbf{y}} \quad \text{where } N = |\mathbf{N}| \text{ magnitude of force exerted by the plane} \\
\mathbf{F_g} &= F_g \sin\theta\hat{\mathbf{x}} - F_g \cos\theta\hat{\mathbf{y}} \\
\text{where } F_g &= mg = \text{ force due to gravity, acceleration } g = 9.8 \text{ m/sec}^2
\end{aligned}$$

Step 3: write down Newton's laws to get the Equations of Motion (EOM):

$$\begin{aligned}
\text{Newton II: force } \mathbf{F} &= \dot{\mathbf{p}} = \frac{d}{dt}(m\dot{\mathbf{r}}) = m\ddot{\mathbf{r}} \\
\text{where} \quad \mathbf{r} &= x\hat{\mathbf{x}} = \text{ block's position vector} \\
\mathbf{F} &= \mathbf{F_g} + \mathbf{N} = F_g \sin\theta\hat{\mathbf{x}} + (N - F_g\cos\theta)\hat{\mathbf{y}}
\end{aligned}$$

Step 4: write the x & y components of the EOM:

$$\begin{aligned}
\hat{\mathbf{x}} &: \quad mg\sin\theta = m\ddot{x} \\
\hat{\mathbf{y}} &: \quad N - F_g\cos\theta = 0 \quad \text{since the block is confined to plane, ie, } \ddot{y} = 0
\end{aligned}$$

Step 5: solve:

the acceleration (1) is $\mathbf{a} = \ddot{\mathbf{r}} = \ddot{x}\hat{\mathbf{x}} = g\sin\theta\hat{\mathbf{x}}$

integrate to get the block's velocity:

$$\begin{aligned}
\ddot{x} &= \frac{d^2x}{dt} = \frac{dv}{dt} = g\sin\theta \\
\text{integrate (2): } v(t) &= gt\sin\theta + \text{ constant} \\
\text{choose initial conditions: } \quad x(0) &= 0 \quad \text{and } v(0) = 0 \Rightarrow \text{ constant} = 0
\end{aligned}$$

How do I solve for (3) v(t)?

$$\begin{aligned}
\text{integrate again:} \quad x(t) &= \frac{1}{2}gt^2\sin\theta \\
\text{so} \quad t &= \sqrt{2x/g\sin\theta} \\
\text{(3) insert into } v(t) \text{ to get } v(x)\text{:} \quad v(x) &= g\sin\theta\sqrt{2x/g\sin\theta} = \sqrt{2gx\sin\theta}
\end{aligned}$$

Another Example: problem 2–51

A boat of mass m glides with initial velocity v_0, and is slowed by a viscous force bv^2 where $b =$ constant.

What are the boat's velocity $v(t)$ and position $x(t)$?

Apply Newton's law:

$$F = -bv^2 = m\ddot{x} = m\frac{dv}{dt}$$

How do I solve this for $v(t)$?

$$\begin{aligned}
\text{write} \quad \frac{dv}{v^2} &= -\frac{b}{m}dt \\
\text{integrate:} \quad \int_{v_0}^{v(t)} v'^{-2}dv' &= -\frac{b}{m}\int_0^t dt' \\
-\left.\frac{1}{v'}\right|_{v_0}^{v} &= -\frac{1}{v} + \frac{1}{v_0} = -\frac{bt}{m} \\
\frac{1}{v} &= \frac{1}{v_0} + \frac{bt}{m} \\
\text{or} \quad v(t) &= \frac{v_0}{1 + btv_0/m}
\end{aligned}$$

To get $x(t)$, use $v = dx/dt$ and integrate $dx = vdt$:

$$\begin{aligned}
x(t) &= \int_0^{x(t)} dx' = \int_0^t \frac{v_0 dt'}{1 + bv_0 t'/m} \\
\text{do } u \text{ substitution:} \quad u &= 1 + bv_0 t'/m \quad \text{so } du = bv_0 dt'/m \\
\text{and} \quad x(t) &= \frac{m}{b}\int_1^{1+bv_0t/m} \frac{du}{u} = \frac{m}{b}\ln(1 + bv_0 t/m)
\end{aligned}$$

Problem Set #1
due Thursday Sept. 22
at start of class

text problems: 2–3, 2–15, 2–16, 2–21, 2–28, 2–53.

Conservation Theorems

These follow from Newton's laws, and are are quite handy since they are additional constraints that can be used to simplify the problem at hand.

Linear momentum $\mathbf{p}$ is conserved by a free particle for which $\mathbf{F} = 0$.
This follows from law II., $\mathbf{F} = \dot{\mathbf{p}} = 0 \Rightarrow \mathbf{p} =$ constant.

Angular momentum $\mathbf{L}$ is conserved when torque $\mathbf{N} = 0$.
Recall from PHY 305 that

$$\begin{aligned} \mathbf{L} &= \mathbf{r} \times \mathbf{p} \quad \text{particle's angular momentum} \\ \mathbf{N} &= \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \dot{\mathbf{p}} \quad \text{torque} \\ \text{so} \quad \dot{\mathbf{L}} &= \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} = \mathbf{N} \\ \text{thus} \quad \mathbf{L} &= \text{constant when } \mathbf{N} = 0 \end{aligned}$$

Energy

Work = energy required to change a system's state.
The work done on a particle by force $\mathbf{F}$ while moving from position $\mathbf{r_1} \rightarrow \mathbf{r_2}$:

$$W \; = \; \int_{\mathbf{r_1}}^{\mathbf{r_2}} \mathbf{F} \cdot \mathbf{dr}$$

$$\begin{aligned}
\text{Note that} \quad \mathbf{F} \cdot \mathbf{dr} \; &= \; m\ddot{\mathbf{r}} \cdot \mathbf{dr} = m\ddot{\mathbf{r}} \cdot \frac{\mathbf{dr}}{\mathbf{dt}} dt = m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} dt \\
&= \; \frac{1}{2} m \frac{d}{dt}(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) dt = \frac{1}{2} m \frac{dv^2}{dt} dt \\
\text{So} \quad W \; &= \; \frac{1}{2} m \int_{\mathbf{r_1}}^{\mathbf{r_2}} dv^2 = \frac{1}{2} m (v_2^2 - v_1^2) \\
W \; &= \; T_2 - T_1 \\
\text{where} \quad T_i \; &= \; \frac{1}{2} m v_i^2 = \; \text{particle's Kinetic Energy (KE) at position } \mathbf{r_1}
\end{aligned}$$

Potential Energy (PE) = stored energy;
the capacity of a particle to do work later on.

This time assume that the work done by force $\mathbf{F}$ obeys

$$\begin{aligned}
W \; &= \; \int_{\mathbf{r_1}}^{\mathbf{r_2}} \mathbf{F} \cdot \mathbf{dr} = -(U_2 - U_1) \\
\text{where} \quad U_i \; &= \; U(\mathbf{r_i}) = \; \text{some function of particle's position } \mathbf{r_i} \\
&= \; \text{the particle's PE}
\end{aligned}$$

Note that the relationship $\mathbf{F} = -\nabla U$ satisfies the above,

$$\begin{aligned}
\text{where} \quad \nabla &= \text{the gradient operator} \\
\text{According to Eqn. (1.116)} \quad \nabla &= \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}} \text{ in Cartesian coord's} \\
\text{so} \quad F_x &= -\frac{\partial U}{\partial x} \text{ is the x–component of the force,} \\
F_y &= -\frac{\partial U}{\partial y} \text{ the y–component, etc}
\end{aligned}$$

$$\text{Thus} \quad W = -\int_{\mathbf{r_1}}^{\mathbf{r_2}} (\nabla U)\cdot d\mathbf{r}$$

Now evaluate $\nabla U \cdot d\mathbf{r}$:

Keep in mind that the particle's position vector depends on time:
$\mathbf{r} = \mathbf{r}(t)$, so $U(\mathbf{r}) = U(x(t), y(t), z(t))$ in Cartesian coordinates, and

$$\begin{aligned}
\frac{dU}{dt} &= \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt} + \frac{\partial U}{\partial t} \quad \text{by Chain Rule} \\
\text{so} \quad \frac{dU}{dt} &= (\nabla U) \cdot \dot{\mathbf{r}} \\
\Rightarrow \quad dU &= (\nabla U)\cdot d\mathbf{r} \\
\text{so} \quad W &= -\int_{\mathbf{r_1}}^{\mathbf{r_2}} dU = -(U_2 - U_1)
\end{aligned}$$

Evidently the work done on the particle by $\mathbf{F}$ is
the change in the particle's KE $= -1\times$ change in PE:

$$\begin{aligned}
W &= T_2 - T_1 = -(U_2 - U_1) \\
\text{if} \quad E_i &= T_i + U_i = \text{ particle's total energy while at } \mathbf{r_i}, \\
\text{then} \quad E_2 &= T_2 + U_2 = T_1 + U_1 = E_1
\end{aligned}$$

This indicates that the particle's energy $E = T + U$ is conserved.

A *conservative system* is one where the forces are obtained from the system's potential energy U that is a function of *position only*: $U = U(\mathbf{r})$

If U has an explicit time dependence, or if U has a dependence on the particle's velocity (perhaps due to friction), the system is not conservative, and $E \neq$ constant.

Since $\mathbf{F} = -\nabla U$, adding a constant to U doesn't alter the physics, ie, $\mathbf{F}$.

Thus we need only measure U relative to some reference value $U_{ref} = U(\mathbf{r}_{ref})$.

Common reference points are at $\mathbf{r}_{ref} = 0$ or $\mathbf{r}_{ref} = \infty$

From our earlier definition of work work:

$$\begin{aligned} W &= -\int_{\mathbf{r_{ref}}}^{\mathbf{r}} dU = -[U(\mathbf{r}) - U(\mathbf{r_{ref}})] \\ \text{so} \quad U(\mathbf{r}) &= U(\mathbf{r_{ref}}) - W = U_{ref} - \int_{\mathbf{r_{ref}}}^{\mathbf{r}} \mathbf{F} \cdot \mathbf{dr} \end{aligned}$$

Since the value of U_{ref} is unimportant, we usually set $U_{ref} = 0$,

$$\text{so the system's PE is} \quad U = -\int_{\mathbf{r_{ref}}}^{\mathbf{r}} \mathbf{F} \cdot \mathbf{dr}$$

Trivial problem: Shoot a gun.
The bullet's velocity is v. What is the work done on the bullet by the gun?
$W = T_2 - T_1 = \frac{1}{2}mv^2$.

Another simple example:
Lift a brick to a height $y = h$. What work did you just do?
Since $\mathbf{F} = -mg\hat{\mathbf{y}} = -\nabla U = -\frac{dU}{dy}\hat{\mathbf{y}}$, the brick's PE is $U(y) = mgy$,
so the work done on the brick is $W = -(U_2 - U_1) = -mgh$,
with the sign indicating that you *lost* energy putting up the brick.

Graphical Analysis of a system's PE

Can provide a qualitative understanding of a particle's trajectory without actually doing any calculations.

Consider the 1D potential $U(x)$ plotted in Fig. 2-14 in the text:

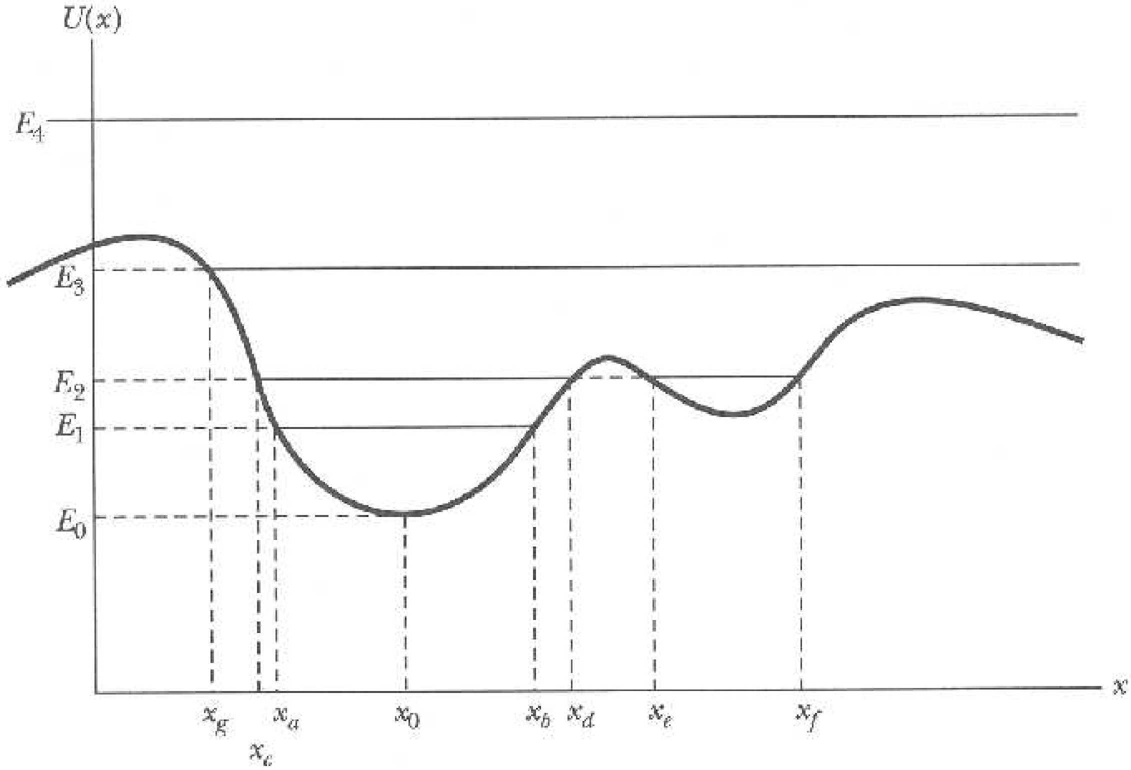


FIGURE 2-14 Potential energy $U(x)$ curve with various energies E indicated. For certain energies, for example E_1 and E_2, the motion is bounded.

Recall that a particle's total energy is $E = T + U$.

Suppose a particle has energy E_0. What range of motions (ie, position x and velocity v) might the particle exhibit?

When $E = E_0$, $T = 0$, $v = 0$, $U = E_0$, and $x = x_0$
$\Rightarrow$ the particle is stationary.

What if the particle has $E = E_1$? E_2? E_3? E_4?

Equilibrium Points & Stability

An equilibrium point is a site is a site $\mathbf{r}_{eq}$ where the force is zero, ie, where $\mathbf{F} = -\nabla U(\mathbf{r}_{eq}) = 0$ or $U(\mathbf{r}_{eq})$ is flat.

For a 1D system, solve $\frac{\partial U}{\partial x}|_{x=x_{eq}} = 0$ to determine the equilibrium point x_{eq}.

However that equilibrium point might be stable (like a marble resting at the bottom of a bowl), or unstable (like a pin standing on its point).

Test the stability of an equilibrium point:
Put the particle at the equilibrium point $\mathbf{r} = \mathbf{r}_{eq}$, and jiggle it, (e.g., displace it slightly or give it a small velocity kick).

Does the particle oscillate around $\mathbf{r} = \mathbf{r}_{eq}$?
$\Rightarrow$equilibrium point is stable.

Or does the particle 'roll away'?
$\Rightarrow$unstable equilibrium point.

To answer this, do a *linear stability analysis*:
Consider a 1D potential $U = U(x)$ and Taylor expand about the equilibrium point $x = x_{eq}$:

$$U(x) \;=\; U(x_{eq}) + (x - x_{eq})\left(\frac{\partial U}{\partial x}\right)_{x_{eq}} + \frac{1}{2}(x - x_{eq})^2\left(\frac{\partial^2 U}{\partial x^2}\right)_{x_{eq}} + \cdots$$

The first term is an unimportant constant,
the second is zero since $\partial U/\partial x|_{x_{eq}} = 0$, so

$$U(x) \simeq \frac{1}{2}k(x - x_{eq})^2 \quad \text{where} \quad k \equiv \left(\frac{\partial^2 U}{\partial x^2}\right)_{x_{eq}}$$

Note that the potential near an equilibrium point resembles that of a spring since $F = -dU/dx = -k(x - x_{eq})$, which is Hook's law for a spring force.

Stability requires $k = \left(\frac{\partial^2 U}{\partial x^2}\right)_{x_{eq}} > 0$, which results in a restoring force that opposes any small displacements from $x = x_{eq}$.

In other words, $U(x)$ has a local minimum at the stable equilibrium point, and a particle that is displaced slightly from equilibrium simply oscillates about $x = x_{eq}$.

Instability occurs when $k < 0$.
The particle is sitting on a local maxima in $U(x)$, and any infinitesimal displacement from equilibrium results in a force that pushes the particle even further away.

However if $k = 0$, the next higher–order term in the expansion for $U(x)$ needs to be examined.

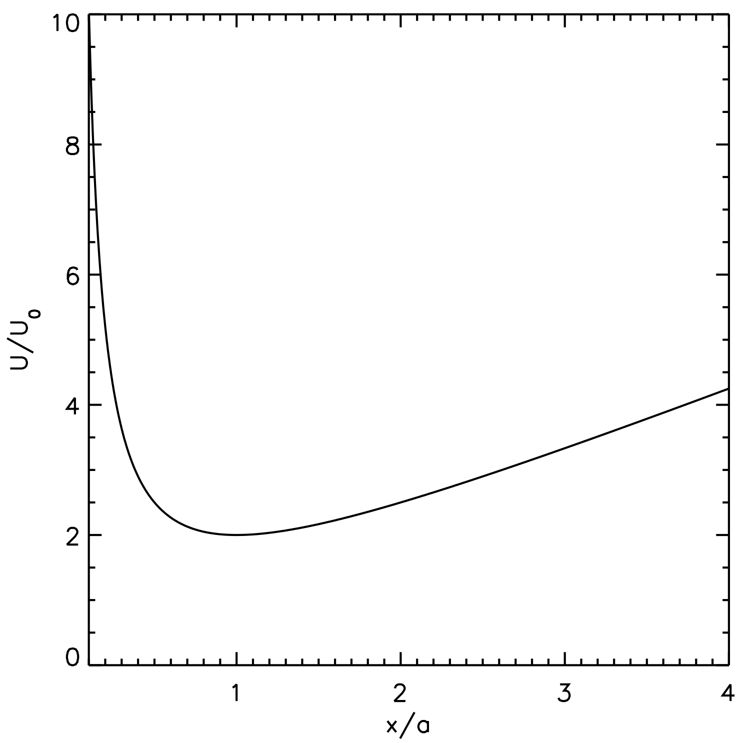
Example: problem 2–47

A particle moves in the $x > 0$ region of the potential

$$U(x) = U_0 \left(\frac{a}{x} + \frac{x}{a} \right) \quad \text{where } U_0, a > 0.$$

Find any equilibrium points, and check their stability.

1. Plot $U(x)$:



Where is the equilibrium site? How do I derive its location?

The equilibrium site x_{eq} is where the force is zero, or where

$$\left.\frac{\partial U}{\partial x}\right|_{x_{eq}} = U_0 \left(-\frac{a}{x_{eq}^2} + \frac{1}{a} \right) = 0$$

$$\Rightarrow x_{eq} = a \quad \text{is the equilibrium point}$$

$$\text{check stability:} \quad k = \left.\frac{\partial^2 U}{\partial x^2}\right|_{x_{eq}} = \frac{2U_0 a}{x_{eq}^3} = \frac{2U_0}{a^2} > 0.$$

So is x_{eq} a stable or unstable equilibrium site?